DFT

Discrete Fourier Transform

DFT

It is a finite duration discrete freq sequence which is obtained by sampling one period of fourier transform. Sampling is done at 'N' equally spaced points, over the period extending from $\omega = 0$ to $\omega = 2\pi$.

Mathematical Eq'n

The DFT of discrete sequence $x(n)$ is denoted by $X(k)$. It is given by

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}}$$

Inverse DFT

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{\frac{j2\pi kn}{N}}$$

$$W_N = e^{\frac{-j2\pi}{N}}$$ twiddle factor

Twiddle factor makes the computation of DFT a bit easy and fast

$$X(K) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$$

$$x(n) = \sum_{n=0}^{N-1} X(k)\, W_N^{-kn}$$

DFT of Standard Signals ⇒

Unit Impulse fn $\delta(n)$

$$X(K) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi kn/N}$$

$\delta(n) = 1$ only at $n=0$

$X(K) = \delta(0)\, e^{0} = 1$

$\therefore \delta(n) = 1$

Obtain DFT of delayed unit impulse $\delta(n-n_0)$

$x(n) = \delta(n-n_0)$

$$X(K) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi kn/N}$$

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Wk	M	T	W	T	F	S	S
18					1	2	3
19	4	5	6	7	8	9	10
20	11	12	13	14	15	16	17
21	18	19	20	21	22	23	24
22	25	26	27	28	29	30	31

$\delta(n-n_0) = 1$ only at $n=n_0$

$X(k) = 1 \cdot e^{-j2\pi k n_0/N}$

$\delta(n-n_0) = e^{-j2\pi k n_0/N}$

$\delta(n+n_0) = e^{j2\pi k n_0/N}$

Obtain N point DFT of exponential sequence

$x(n) = a^n u(n)$ for $0 \leq n \leq N-1$

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}$$

$x(n) = a^n u(n)$

$$X(k) = \sum_{n=0}^{N-1} a^n e^{-j2\pi kn/N}$$

$$X(k) = \sum_{n=0}^{N-1} \left(a e^{-j2\pi \frac{k}{N}}\right)^n$$

$$\sum_{k=N_1}^{N_2} A^k = \frac{A^{N_1} - A^{N_2+1}}{1-A}$$

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Wk	M	T	W	T	F	S	S
14			1	2	3	4	5
15	6	7	8	9	10	11	12
16	13	14	15	16	17	18	19
17	20	21	22	23	24	25	26
18	27	28	29	30			

$N_1 = 0$ $N_2 = N-1$

$A = ae^{-j2\pi k/N}$

$$X(K) = \frac{\left(ae^{-j2\pi k/N}\right)^0 - \left(ae^{-j2\pi k/N}\right)^{N-1+1}}{1 - ae^{-j2\pi k/N}}$$

$$X(K) = \frac{1 - a^N e^{-j2\pi K}}{1 - ae^{-j2\pi k/N}}$$

Using Euler's identity to the numerator term

$$e^{-j2\pi K} = \cos 2\pi k - j\sin 2\pi K$$

But K is an integer

$\therefore \cos 2\pi K = 1$ and $\sin 2\pi k = 0$

$$e^{-j2\pi k} = 1 - j0 = 1$$

$$X(K) = \frac{1 - a^N}{1 - ae^{-j2\pi k/N}}$$

$a^n u(n) \longleftrightarrow$

Compute DFT of sequence given as $x(n) = (-1)^n$ for

$N = 4$

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18					1	2	3
19	4	5	6	7	8	9	10
20	11	12	13	14	15	16	17
21	18	19	20	21	22	23	24
22	25	26	27	28	29	30	31

The range of n is $n=0$ to $N-1$

that means $n=0$ to 3

Given sequence $x(n) = (-1)^n$

$n=0 \Rightarrow x(0) = (-1)^0 = 1$

$n=1 \Rightarrow x(1) = (-1)^1 = -1$

$n=2 \Rightarrow x(2) = (-1)^2 = 1$

$n=3 \Rightarrow x(3) = (-1)^3 = -1$

$$x(n) = \{\underset{\uparrow}{1}, -1, 1, 1\}$$

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j2\pi kn/N}$$

$$X(k) = \sum_{n=0}^{3} x(n)\, e^{-j2\pi kn/4}$$

$$X(k) = \sum_{n=0}^{3} x(n)\, e^{-j\pi kn/2}$$

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Wk	M	T	W	T	F	S	S
14			1	2	3	4	5
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16	13	14	15	16	17	18	19
17	20	21	22	23	24	25	26
18	27	28	29	30			

For $k=0$ $\quad X(0) = \sum_{n=0}^{3} x(n) e^{0} = \sum_{n=0}^{3} x(n)$

$X(0) = x(0) + x(1) + x(2) + x(3)$

$1 - 1 + 1 - 1 = 0$

For $k=1$

$X(1) = \sum_{n=0}^{3} x(n) e^{-j\pi n/2}$

$X(1) = x(0) e^{0} + x(1) e^{-\frac{j\pi}{2}} + x(2) e^{-j\pi} + x(3) e^{-j\pi 3/2}$

$X(1) = 1 - 1(-0 - j1) + 1(-1 - j0) - (0 + j1)$

$X(1) = 0$

For $k=2$

$X(2) = \sum_{n=0}^{3} x(n) e^{-j\pi n}$

$X(2) = x(0) e^{0} + x(1) e^{-j\pi} + x(2) e^{-j\pi 2} + x(3) e^{-j3\pi}$

$X(2) = 1 - (-1 - j0) + x(1)(1 - j0) - 1(-1 - j0)$

$X(2) = 4$

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WK	M	T	W	T	F	S	S
18					1	2	3
19	4	5	6	7	8	9	10
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For K=3

$$X(3) = \sum_{n=0}^{3} x(n)\, e^{-j3\pi n/2}$$

$$X(3) = x(0)e^{0} + x(1)e^{-3\pi/2} + x(2)e^{-j3\pi} + x(3)\, e^{-j9\pi/2}$$

$$X(3) = 1 - (0+j1) + (-1-j0) - (0-j1)$$

$$X(3) = 0$$

$$X(K) = \{0, 0, 4, 0\}$$

Obtain the Value of $x(4)$ for 8 pt DFT if $x(n) = \{1, -1, 0, 2, 1, -2, -1, 1\}$

$$X(K) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi K n}{N}}$$

$N = 8$

$$\therefore X(K) = \sum_{n=0}^{7} x(n)\, e^{\frac{-j2\pi kn}{8}}$$

$X(4)$ means the here to put $K=4$

$$X(4) = \sum_{n=0}^{7} x(n)\, e^{\frac{-j2\pi \times 4n}{8}} = \sum_{n=0}^{7} x(n)\, e^{-j\pi n}$$

$$X(4) = x(0)e^{0} + x(1)e^{-j\pi} + x(2)e^{-j2\pi} + x(3)e^{-j3\pi} + x(4)e^{-j4\pi} + x(5)e^{-j5\pi} + x(6)e^{-j6\pi} + x(7)e^{-7j\pi}$$

$$X(4) = 1+1-2+1+2-1-1$$

$$\boxed{\therefore X(4) = 1}$$

Determine & sketch the Fourier transform

$$x(n) = \{1, 1, \underset{\uparrow}{1}, 1, 1\}$$

$$X(K) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}}$$

Here $N=5$ and given sequence is having the range $n = -2$ to 2

$$X(K) = \sum_{n=0}^{4} x(n) e^{\frac{-j2\pi Kn}{5}}$$

For $K=0$ = $x(0) = 5$

For $K=1$ = $X(1) = 0$

$K=2$ = $X(2) = 0$

$K=3$ = $X(3) = 0$

$K=4$ = $X(4) = 0$

DFT as Linear Transformation Matrix

Twiddle factor $W_N = e^{-j2\pi/N}$

$x(n)$ can be denoted by x_N

$$x_N = \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \\ x(N-1) \end{bmatrix}_{N\times 1}$$

$$X(K) = \sum_{n=0}^{N-1} x(n) W_N^{Kn}$$

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Wk	M	T	W	T	F	S	S
14			1	2	3	4	5
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17	20	21	22	23	24	25	26
18	27	28	29	30			

Our greatest battles are that with our own minds.

W_N^{kn} in Matrix form

$$W_N^{kn} = \begin{array}{c} \\ k=0 \\ k=1 \\ | \\ k=N-1 \end{array} \begin{array}{c} n=0 \quad\quad n=1 \quad\quad \cdots \quad n=N-1 \\ \begin{bmatrix} W_N^0 & W_N^0 & \cdots & W_N^0 \\ W_N^0 & W_N^1 & \cdots & \\ | & & & \\ W_N^0 & W_N^{N-1} & W_N^{2(N-1)} & W_N^{(N-1)^2} \end{bmatrix} \end{array}$$

$$X_N = [W_N] x_N$$

Similarly IDFT

$$x_N = \frac{1}{N} [W_N^*] X_N.$$

determine 2pt and 4pt DFT of sequence

$x(n) = u(n) - u(n-2)$

$x(n) = \{1, 1\}$

2pt

$N = 2$

$$W_N = e^{-j\frac{2\pi}{N}} = e^{-j\frac{2\pi}{2}} = e^{-j\pi}$$

$$W_N^{kn} = e^{-j\pi kn}$$

$$W_2^{kn} = \begin{array}{c|cc} & n=0 & n=1 \\ \hline k=0 & W_2^0 & W_2^0 \\ k=1 & W_2^0 & W_2^1 \end{array}$$

$kn=0 \qquad W_2^0 = e^{-j\pi \times 0} = e^0 = 1$

$kn=1 \qquad W_2^1 = e^{-j\pi} \quad e^{-j\pi} \cos\pi - j\sin\pi = -1$

$$W_2^{kn} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$X_N = x(n) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$X_N = [W_N] \cdot x_N$$

$$X_K = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} (1\times1) + (1\times1) \\ (1\times1) + (1\times-1) \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$x(K) = \{2, 0\}$$

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Wk	M	T	W	T	F	S	S
14			1	2	3	4	5
15	6	7	8	9	10	11	12
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17	20	21	22	23	24	25	26
18	27	28	29	30			

$$W_N = W_4 = e^{-j2\frac{\pi}{4}} = e^{-j\pi/2}$$

$$W_N^{kn} = e^{-j\pi/2\, kn}$$

$$[W_4] = W_4^{kn} = \begin{array}{c|cccc} & n=0 & n=1 & n=2 & n=3 \\ k=0 & W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ k=1 & W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ k=2 & W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ k=3 & W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{array}$$

$W_4^0 = 1$

$W_4^1 = e^{-j\pi/2} = \cos\pi/2 - j\sin\pi/2 = -j$

$W_4^2 = e^{-j\pi/2 \cdot 2} = e^{-j\pi} = -1$

$W_4^3 = e^{-j\frac{\pi}{2}\times 3} = \cos 3\pi/2 - j\sin 3\pi/2 = +j$

According to cyclic Property

$W_4^0 = W_4^4 = 1$

$W_4^1 = W_4^5 = W_4^9$

$W_4^2 = W_4^6 = W_4^{10}$

$W_4^3 = W_4^7 = W_4^{11}$

$$[W_4] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & -j \end{bmatrix}$$

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08

$X_K = [W_N] \cdot x_N$

09

10

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

11

12

01

$X_4 = \begin{bmatrix} 2 \\ 1-j \\ 0 \\ 1+j \end{bmatrix}$

02

03

04

Calculate 8 pt DFT

05

$x(n) = \{1, 2, 1, 2, 0, 0, 0, 0\}$

06

$W_8^{kn} = e^{-j\frac{\pi}{4}kn}$

07

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0.707-j0.707 & -j & -0.707-j0.707 & -1 & -0.707+j0.707 & j & 0.707+j0.707 \\ 1 & -j & -1 & j & 1 & -j & -1 & j \\ 1 & -0.707-j0.707 & j & 0.707-j0.707 & -1 & 0.707+j0.707 & -j & -0.707+j0.707 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -0.707+j0.707 & -j & 0.707+j0.707 & -1 & 0.707-j0.707 & j & -0.707-j0.707 \\ 1 & j & -1 & -j & 1 & j & -1 & -j \\ 1 & 0.707+j0.707 & j & -0.707+j0.707 & -1 & -0.707-j0.707 & -j & 0.707-j0.707 \end{bmatrix}$$

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Wk	M	T	W	T	F	S	S
18					1	2	3
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Properties of DFT ⇒

Linearity →

If $x_1(n) — X_1(K)$

$x_2(n) — X_2(K)$

$a_1 x_1(n) + a_2 x_2(n) \xleftrightarrow[N]{} a_1 X_1(K) + a_2 X_2(K)$

Proof ⇒ According to the definition of DFT

$$X(K) = \sum_{n=0}^{N-1} x(n) W_N^{kn}$$

$$x(n) = a_1 x_1(n) + a_2 x_2(n)$$

$$\therefore X(K) = \sum_{n=0}^{N-1} a_1 x_1(n) + a_2 x_2(n) W_N^{kn}$$

$$= \sum_{n=0}^{N-1} a_1 x_1(n) W_N^{kn} + \sum_{n=0}^{N-1} a_2 x_2(n) W_N^{kn}$$

$$X(K) = a_1 X_1(K) + a_2 X_2(K)$$

Periodicity $\Rightarrow$

If $x(n) — X(K)$

$X(K+N) = X(K)$ for all K

Proof

$$X(K) = \sum_{n=0}^{N-1} x(n) W_N^{Kn}$$

Replace K by K+N

$$X(K+N) = \sum_{n=0}^{N-1} x(n) W_N^{(K+N)n}$$

$$= \sum_{n=0}^{N-1} x(n) W_N^{Kn} \cdot W_N^{Nn}$$

$$W_N^{Nn} = e^{-\frac{j2\pi Nn}{N}} \quad e^{-j2\pi n}$$

$\cos 2\pi n - j \sin 2\pi n = 1$

$$X(K+N) = \sum_{n=0}^{N-1} x(n) W_N^{Kn}$$

$X(K+N) = X(K)$

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Circular Symmetries of a Sequence ⇒

Periodic sequence is denoted by $x_p(n)$

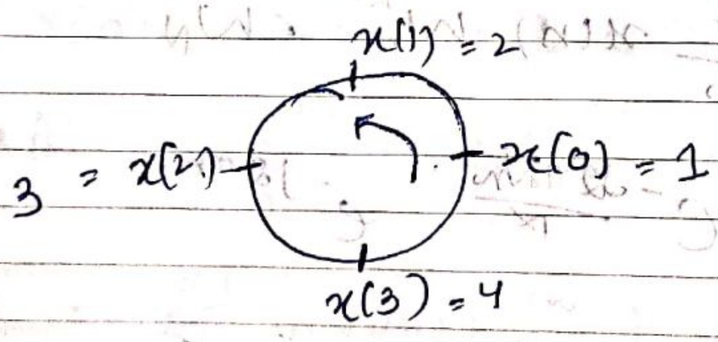
Period of $x_p(n)$ is N which means

after N the sequence x(n) repeats itself.

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n - lN)$$

$$x'(n) = x((n-2))_4$$

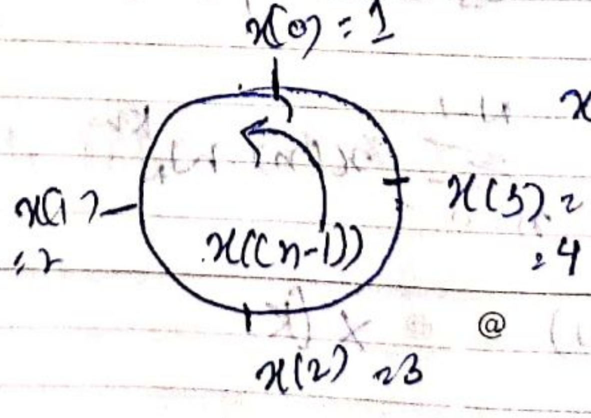
$$x(n) = \{1, 2, 3, 4\}$$



Anticlock

$x((n))_4$

Delay



$x((n-1))$

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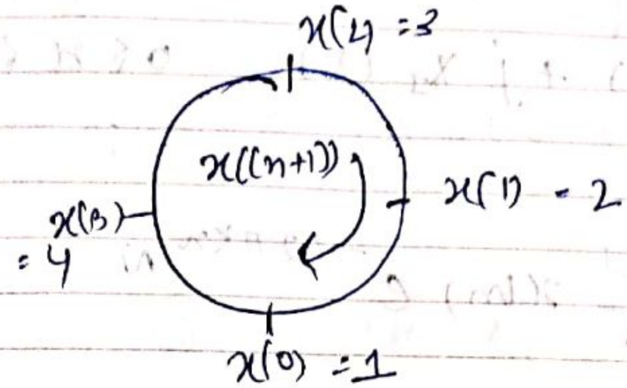
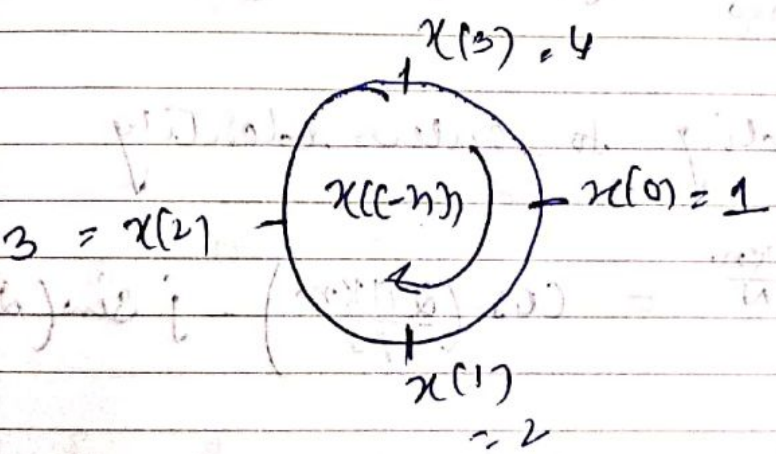
Wk	M	T	W	T	F	S	S
14			1	2	3	4	5
15	6	7	8	9	10	11	12
16	13	14	15	16	17	18	19
17	20	21	22	23	24	25	26
18	27	28	29	30			

Advance $x((n+1))$



Clockwise

Folded



Circularly even $= x(N-n) = x(n)$.

Circularly odd $= x(N-n) = -x(n)$

Symmetry Properties ⟹

We know that DFT of sequence $x(n)$ is denoted by $X(k)$ now if $x(n)$ and $X(k)$ are complex valued sequence then it can be represented as

$x(n) = x_R(n) + j x_I(n) \quad 0 \leq n \leq N-1$ — (1)

$X(k) = X_R(k) + j X_I(k) \quad 0 \leq n \leq N-1$ — (2)

Defination

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}$$

$$X(k) = \sum_{n=0}^{N-1} x_R(n) + j x_I(n)] e^{-j2\pi kn/N}$$

But according to euler identity

$$e^{-j\frac{2\pi kn}{N}} = \cos\left(\frac{2\pi kn}{N}\right) - j \sin\left(\frac{2\pi kn}{N}\right)$$

$$X(k) = \sum_{n=0}^{N-1} [x_R(n) + j x_I(n)]] \cos\frac{2\pi kn}{N} - j \sin\frac{2\pi kn}{N}$$

$$\therefore X(k) = \sum_{n=0}^{N-1} [X_R(n) \cdot \cos\frac{2\pi kn}{N} - j X_R \sin\frac{2\pi kn}{N}]$$

$$+ j X_I \, x(n) \cos\frac{2\pi kn}{N} - j^2 X_I(n) \sin\left(\frac{2\pi kn}{N}\right]$$

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Wk	M	T	W	T	F	S	S
14			1	2	3	4	5
15	6	7	8	9	10	11	12
16	13	14	15	16	17	18	19
17	20	21	22	23	24	25	26
18	27	28	29	30			

$j^2 = -1$

$$X(k) = \sum_{n=0}^{N-1} \left[x_R(n) \cos\left(\frac{2\pi kn}{N}\right) + x_I(n) \sin\left(\frac{2\pi kn}{N}\right)\right]$$

$$-j \sum_{n=0}^{N-1} \left[x_R(n) \sin\left(\frac{2\pi kn}{N}\right) - x_I(n) \cos\left(\frac{2\pi kn}{N}\right)\right] \quad \text{—} (3)$$

Compare eq (3) with eq (2)

$$X_R(k) = \sum_{n=0}^{N-1} \left[x_R(n) \cos\left(\frac{2\pi kn}{N}\right) + x_I(n) \sin\left(\frac{2\pi kn}{K}\right)\right] \quad \text{—} (4)$$

$$X_I(k) = -\sum_{n=0}^{N-1} \left[x_R(n) \sin\left(\frac{2\pi kn}{N}\right) - x_I \cos\left(\frac{2\pi kn}{N}\right)\right] \quad \text{—} (5)$$

Similarly Real & Imagi part of IDFT

$$x_R(n) = \frac{1}{N} \sum_{k=0}^{N-1} \left[X_R(k) \cos\left(\frac{2\pi kn}{N}\right) - X_I(k) \sin\left(\frac{2\pi kn}{N}\right)\right]$$

$$x_I(n) = \frac{1}{N} \sum_{k=0}^{N-1} \left[X_R(k) \sin\left(\frac{2\pi kn}{N}\right) + X_I(k) \cos\left(\frac{2\pi kn}{N}\right)\right]$$

When $x(n)$ is real valued

$$X(N-k) = X(-k) = X^*(k)$$

Proof

$$X(K) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$$

K by N-K

$$X(N-K) = \sum_{n=0}^{N-1} x(n)\, W_N^{(N-k)n}$$

$$= \sum_{n=0}^{N-1} x(n)\, W_N^{Nn} \cdot W_N^{-kn}$$

$$X(N-K) = \sum_{n=0}^{N-1} x(n)\, W_N^{-kn}$$

$$\therefore X(-K) = X^*(K)$$

When $x(n)$ is real is even

$x(n) = x(N-n)$

$X(K) = X_R(k)$

Proof

$$X_R(K) = \sum_{n=0}^{N-1} x_R(n) \cos\left(\frac{2\pi kn}{N}\right)$$

The 1st five point of the 8pt DFT of a real valued sequence are {0.25, 1.25 − j0.3018, 0, 0.125 − j0.0518, 0} Determine the remaining three pts

Soln

$x(0) = 0.25$
$x(1) = 0.125 - j0.3018$
$x(2) = 0$
$x(3) = 0.125 - j0.0518$
$x(4) = 0$

According to symmetry property

$X^*(k) = X(N-k)$

$X(K) = X^*(N-k)$

$X(k) = X^*(8-k)$

$X(5) = X^*(8-5) = X^*(3)$

$X^*(3) = 0.125 + j0.0518$

$X^*(6) = 0$

$X(7) = 0.125 + j0.3018$

Circular Convolution

If $x_1(n) \longrightarrow X_1(K)$

$x_2(n) \longrightarrow X_2(K)$

$x_1(n) \circledast x_2(n) \longrightarrow X_1(K) \cdot X_2(K)$

$\circledast$ indicates circular convolution

$$y(m) = \sum_{n=0}^{N-1} x_1(n) \cdot x_2((m-n))_N$$

$x_2((m-n))_N$ indicates the circular convolution

Proof

$$X_1(K) = \sum_{n=0}^{N-1} x_1(n)\, e^{-\frac{j2\pi Kn}{N}} \quad \text{--- (1)}$$

$$X_2(K) = \sum_{l=0}^{N-1} x_2(l)\, e^{-\frac{j2\pi Kl}{N}} \quad \text{--- (2)}$$

$$Y(K) = X_1(K) \cdot X_2(K) \quad \text{--- (3)}$$

Let IDFT of $Y(K)$ be $y(m)$

$$y(m) = \frac{1}{N} \sum_{K=0}^{N-1} Y(K)\, e^{\frac{j2\pi Km}{N}} \quad \text{--- (4)}$$

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Wk	M	T	W	T	F	S	S
23	1	2	3	4	5	6	7
24	8	9	10	11	12	13	14
25	15	16	17	18	19	20	21
26	22	23	24	25	26	27	28
27	29	30					

$$y(m) = \frac{1}{N} \sum_{k=0}^{N-1} x_1(k) \cdot X_2(k)\, e^{\frac{j2\pi km}{N}} \quad \text{—(5)}$$

$$y(m) = \frac{1}{N} \sum_{k=0}^{N-1} \left[\sum_{n=0}^{N-1} x_1(n)\, e^{\frac{-j2\pi kn}{N}} \right] \left[\sum_{l=0}^{N-1} x_2(l)\, e^{\frac{-j2\pi kl}{N}} \right] e^{j\frac{2\pi km}{N}} \quad \text{—(6)}$$

Rearraging the summation

$$y(m) = \frac{1}{N} \sum_{k=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l) \left[\sum_{k=0}^{N-1} e^{\frac{-j2\pi kn}{N}} \cdot e^{-\frac{j2\pi kl}{N}} \cdot e^{\frac{j2\pi km}{N}} \right] \quad \text{—(7)}$$

$$y(m) = \frac{1}{N} \sum_{k=0}^{N-1} x_1(n) \cdot \sum_{l=0}^{N-1} x_2(l) \left[\sum_{k=0}^{N-1} e^{\frac{j2\pi k(m-n-l)}{N}} \right] \quad \text{—(8)}$$

Consider the last term

$$e^{j2\pi k(m-n-l)/N} = \left[e^{j2\pi(m-n-l)/N} \right]^k \quad \text{—(9)}$$

$$\sum_{K=0}^{N-1} a^K = \begin{cases} N & \text{for } a=1 \\ \frac{1-a^N}{1-a} & \text{for } a \neq 1 \end{cases}$$

Let here

$$a = e^{j2\pi \frac{(m-n-l)}{N}}$$

We will consider two Cases :-

Case 1 when $a = 1$

If $m-n-l$ is multiple of N which means

$(m-n-l) = N, 2N, 3N$ ----

$$a = e^{j2\pi \frac{\cancel{N}}{\cancel{N}}} = e^{j2\pi} = 1$$

Case 2 when $a \neq 1$

If $a \neq 1$ that means if $m-n-l$ is not multiple of N then

$$\sum_{K=0}^{N-1} \left(e^{j2\pi \frac{(m-n-l)}{N}}\right)^K = \frac{1-a^N}{1-a}$$

$$\frac{1 - e^{j2\pi \frac{(m-n-l) \cdot \cancel{N}}{\cancel{N}}}}{1 - e^{j2\pi \frac{(m-n-l)}{N}}} \qquad e^{j2\pi(m-n-l)} = 1$$

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Here n, m, l are integers then
$e^{j2\pi(m-n-l)} = 1$

$$y(m) = \frac{1}{N} \sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l)\, N$$

$$y(m) = \sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l)$$

$$y(m) = \frac{1}{N} \sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l) \cdot N$$

$$y(m) = \sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l) \quad \text{—(7)}$$

Now we obtain eq (7) for condition $(m-n-l)$ is multiple of N

$$m - n - l = \pm PN$$

For simplicity we have consider - ve integer

$$l = m - n + PN$$

$$y(m) = \sum_{n=0}^{N-1} x_1(n) \cdot x_2(m-n+PN)$$

If a sequence is periodic and delayed then it can be expressed as

$$x_2(m-n+PN) = x_2((m-n))_N$$

$$y(m) = \sum_{n=0}^{N-1} x_1(n) \cdot x_2((m-n))_N$$

Circular Convolution

① Graphical

② Matrix

③ DFT & IDFT